A Computation Procedure for Reconsideration-Proof Equilibria*

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Abstract

Reconsideration-proof equilibrium is a refinement proposed by Kocherlakota [1996] of subgame perfect equilibrium that applies to infinite horizon settings in which time inconsistency is important. A procedure for computing such equilibria is provided. The procedure is applicable under an assumption about time-separability of the utility function. The class of problems that satisfy the assumption includes four of the five examples in Kocherlakota [1996].

1 Introduction

Reconsideration-proof equilibrium is a refinement proposed by Kocherlakota [1996] of subgame perfect equilibrium that applies to infinite horizon settings in which time inconsistency is important.

To be reconsideration-proof, a strategy must satisfy three properties. First, it must be subgame-perfect. Second, it must have the same continuation value at any histories. Let us call such strategies weakly reconsideration-proof. Accordingly, a continuation value is attached to each weakly reconsideration-proof strategy. Lastly, the strategy must have the highest continuation value among all the weakly reconsideration-proof strategies.

It is not easy to find reconsideration-proof strategies. Though Kocherlakota [1996] provides a characterization of weakly reconsideration-proof strategies, it does not tell us much about how to

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find them. The difficulty in finding reconsideration-proof equilibrium contrasts with that in finding subgame perfect equilibrium, for which the computation algorithm suggested by Abreu et al. [1990] is available.

This paper provides a computation procedure which is applicable under an assumption about time separability of the utility function. The class of problems that satisfy the assumption includes four of the five examples in Kocherlakota [1996].

2 Environment

Time is discrete, and indexed by t. There is an infinitely lived player.

In each period t, there are two subperiods. In the first subperiod, a state variable $z_t \in Z$ is determined, where Z is a set of states. In the second subperiod, knowing the determined value of the state variable z_t, the player chooses an action x_t from a set of actions X.

The environment can be thought of as a reduced form of an environment with a large player and a continuum of agents, as Kocherlakota [1996] showed. The state variable z_t is a choice variable for the continuum of agents. Their choice depends on their expectation about the large player's future actions, $\{\hat{x}_s\}_{s \geq t}$. Here, the choice by the continuum of agents is summarized by a function: the state variable z_t is determined by

$$z_t = \xi\left(\{\hat{x}_s\}_{s \geq t}\right),$$

where ξ is a function from X^∞ to Z.

The utility of the infinitely lived player in period t is given by

$$U(z_t, \{x_s\}_{s \geq t}),$$

where U is a function from $X^\infty \times Z$ to $\mathbb{R}$.

Throughout the paper, the following is assumed:

Assumption 1. *X is a convex and compact subset of a Hausdorff space equipped with topology τ .*

A history is an element of $\mathcal{H} = \bigcup_{t \geq 0} X^t$, where $X^0 = \{\emptyset\}$. A strategy σ is a mapping from $\mathcal{H}$ to X. A continuation path of σ, Φ^σ, is a mapping from $\mathcal{H}$ to X^∞ defined by

$$\Phi^\sigma(h) = \{\sigma(h), \sigma(\langle h, \sigma(h) \rangle), \sigma(\langle h, \sigma(\langle h, \sigma(h) \rangle) \rangle), ...\}$$

for each $h \in \mathcal{H}$. A continuation value of σ, which is denoted by V^σ, is a mapping from $\mathcal{H}$ to $\mathbb{R}$ defined by $V^\sigma(h) = U(\xi(\Phi^\sigma(h)), \Phi^\sigma(h))$ for each $h \in \mathcal{H}$.

3 Solution Concepts and Characterization

3.1 Solution Concepts

First, subgame perfection is defined in the standard way.

Definition 1. A strategy σ is *subgame-perfect* if for any history h, for any $x \in X$,

$$V^\sigma(h) \geq U(\xi(\Phi^\sigma(h)), \langle x, \Phi^\sigma(\langle x, h \rangle) \rangle).$$

Here, since the player at different histories is modeled as different decision makers, unilateral deviations to which subgame perfect strategies should be immune are just deviations in one-period action.

Reconsideration-proofness is defined in the next two definitions, following Kocherlakota [1996].

Definition 2. A subgame-perfect strategy σ is *weakly reconsideration-proof* if there exists $\bar{V}^\sigma \in \mathbb{R}$ such that for any $h \in \mathcal{H}$, $\bar{V}^\sigma = V^\sigma(h)$. Such $\bar{V}^\sigma$ is called the *value* of the weakly reconsideration-proof strategy σ.

Definition 3. A weakly reconsidration-proof strategy σ is *strongly reconsideration-proof* if for any weakly reconsideration-proof strategies σ', $\bar{V}^\sigma \geq \bar{V}^{\sigma'}$.

The weakly reconsideration-proofness requires a necessary condition to be reconsideration-proof. If the condition failed, that is, if a strategy had two continuation strategies that have different continuation values, then, at the history at which the player was supposed to play the continuation strategy with a lower value, he would switch to the continuation strategy with a higher value.

Strongly reconsideration-proof strategies are the best among weakly reconsideration-proof strategies. That implies, if a strategy achieves a higher continuation value at a history, then it is not weakly reconsideration-proof. The player would not switch from a strongly reconsideration-proof strategy in a credible way.

3.2 Characterization

Kocherlakota [1996] gives a characterization of weakly reconsideration-proof strategies. To state it, define

$$
\begin{aligned}
D(V) &= \{\{x_t\}_{t\geq 0} | \text{for each } t \geq 0, U(\xi(\{x_s\}_{s\geq t}), \{x_s\}_{s\geq t}) = V\}, \text{ and} \\
Z(V) &= \{\xi(\{x_t\}_{t\geq 0}) | \{x_t\}_{t\geq 0} \in D(V)\}.
\end{aligned}
$$

Now, the characterization can be stated.

Proposition 1. *[Kocherlakota, 1996] There exists a weakly reconsideration-proof strategy with value V if and only if there exists a subset $D^* \subseteq D(V)$ and a subset $Z^* \subseteq Z(V)$ such that*

1. $Z^* = \xi(D^*)$.

2. *For all $x \in X$ and all $z \in Z^*$, there exists d in D^* such that $U(z, x, d) \leq V$.*

The characterization does not tell much about how to find such pair of D^* and Z^*. In the next section, I specify a subclass of problems and suggest a computation procedure.

4 Class of Problems and Computation Procedure

4.1 Class of Problems

I suggest a procedure for problems that satisfy the following assumption.

Assumption 2. *There exist a constant* β *and a function* $u : X^2 \to \mathbb{R}$ *such that* $U(z_t, \{x_s\}_{s\geq t}) = (1-\beta)\sum_{s=t}^{\infty} \beta^{s-t} u(x_s, x_{s+1})$.

Under the assumption, the characterization of weakly reconsideration-proof strategies can be simplified. The simplified characterization will lead to a procedure. To state it, define

$$\mathcal{X}_0(V) = X,$$

$$\mathcal{X}_1(V) = \{x \in X | \text{there exists } x' \in X \text{ such that } u(x, x') = V\},$$

$$\text{for each } k,\ \mathcal{X}_{k+1}(V) = \{x \in X | \text{there exists } x' \in \mathcal{X}_k(V) \text{ such that } u(x, x') = V\}, \text{ and}$$

$$\mathcal{X}(V) = \bigcap_{k=1}^{\infty} \mathcal{X}_k(V).$$

The next lemma is useful.

Lemma 1. *Under Assumption 2,* $x \in \mathcal{X}(V)$ *if and only if there exists* $\{x_t\}_{t\geq 0} \in D(V)$ *such that* $x_0 = x$.

Proof. See Appendix. □

Now, the simplified characterization is stated as a proposition.

Proposition 2. *Under Assumption 2, there exists a weakly reconsideration-proof strategy with value* V *if and only if for all* $x \in X$*, there exists* x' *in* $\mathcal{X}(V)$ *such that* $u(x, x') \leq V$*.*

Proof. By Proposition 1, it is sufficient to show the equivalence between the following two statements.

1. there exists a subset $D^* \subseteq D(V)$ and a subset $Z^* \subseteq Z(V)$ such that

   (a) $Z^* = \xi(D^*)$.

   (b) For all $x \in X$ and all $z \in Z^*$, there exists d in D^* such that $U(z, x, d) \leq V$.

2. for all $x \in X$, there exists x' in $\mathcal{X}(V)$ such that $u(x, x') \leq V$.

First, since U does not depend on z by Assumption 2, the first statement is equivalent to

$$\text{for all } x \in X, \text{ there exists } d \in D(V) \text{ such that } U(x,d) \leq V.$$

Next, by the time separability of U, it is equivalent to

$$\text{for all } x \in X, \text{ there exists } \{x_t\}_{t\geq 0} \in D(V)$$
$$\text{such that } (1-\beta)u(x,x_0) + \beta U(\{x_t\}_{t\geq 1}) \leq V.$$

Since for any $\{x_t\}_{t\geq 0} \in D(V)$, $U(\{x_t\}_{t\geq 1}) = V$, it is equivalent to

$$\text{for all } x \in X, \text{ there exists } \{x_t\}_{t\geq 0} \in D(V) \text{ such that } u(x,x_0) \leq V.$$

By Lemma 1, it is equivalent to the second statement. □

4.2 Computation Procedure

The computation procedure for reconsideration-proof equilibrium is the following:

1. For each V, compute $\mathcal{X}(V)$ by calculating $\{\mathcal{X}_k(V)\}_{k\geq 1}$.
2. Find the largest V such that for all $x \in X$, there exists x' in $\mathcal{X}(V)$ such that $u(x,x') \leq V$.

Proposition 2 assures that weakly reconsideration-proof strategies achieve the largest V are strongly reconsideration-proof. In the next section, I apply the procedure for the examples in Kocherlakota [1996].

5 Examples

There are five examples in Kocherlakota [1996]. Four of them satisfies Assumption 2. In the following, the procedure is applied to the four examples.

Example 1. (Example 1 in Kocherlakota [1996]) Let $u(x,x') = x - x'$ and $X = [0,1]$. Then,

$$\begin{aligned}\mathcal{X}_1(V) &= \{x \in [0,1] | \text{there exists } x' \in [0,1] \text{ such that } x - x' = V\} \\ &= \{x \in [0,1] | x - V \in [0,1]\} \\ &= [0,1] \cap [V, 1+V],\end{aligned}$$

$$\begin{aligned}\mathcal{X}_k(V) &= \{x \in [0,1] | \text{there exists } x' \in \mathcal{X}_{k-1}(V) \text{ such that } x - x' = V\} \\ &= \{x \in [0,1] | x - V \in \mathcal{X}_{k-1}(V)\} \\ &= [0,1] \cap \bigcap_{k=1} [kV, 1+kV],\end{aligned}$$

and

$$\mathcal{X}(V) = \begin{cases} \emptyset & \text{for } V \neq 0, \\ [0,1] & \text{for } V = 0. \end{cases}$$

Since if $V = 0$, for any $x \in X$, $u(x,x) = 0 = V$, the value of reconsideration-proof equilibrium is $V = 0$. Any strategies that satisfy for each $t \geq 1$, $\sigma(h^{t-1}) = h_{t-1}^{t-1}$ is reconsideration-proof.

Example 2. (Example 2 and 3 in Kocherlakota [1996]) Let $u(x,x') = \sqrt{xx'}$ and $X = [0,1]$. Note that $V \in [0,1]$. Then,

$$\begin{aligned}\mathcal{X}_1(V) &= \{x \in [0,1] | \text{there exists } x' \in [0,1] \text{ such that } \sqrt{xx'} = V\} \\ &= \{x \in [0,1] | \sqrt{x} \in [V, \infty)\} \\ &= [V^2, 1],\end{aligned}$$

$$\begin{aligned}\mathcal{X}_k(V) &= \{x \in [0,1] | \text{there exists } x' \in \mathcal{X}_{k-1}(V) \text{ such that } \sqrt{xx'} = V\} \\ &= \{x \in [0,1] | x - V \in \mathcal{X}_{k-1}(V)\} \\ &= [V^2, 1],\end{aligned}$$

and

$$\mathcal{X}(V) = [V^2, 1].$$

Since if $V = 1$, for any $x \in X$, $u(x, 1) = \sqrt{x} \leq 1 = V$, the value of reconsideration-proof equilibrium is $V = 1$. The strategy that satisfy for each $t \geq 0$, $\sigma(h^{t-1}) = 1$ is reconsideration-proof.

Example 3. (Example 4 in Kocherlakota [1996]) Let $u(x, x') = x - 2x'$ and $X = [0, 1]$. Then,

$$\begin{aligned}\mathcal{X}_1(V) &= \{x \in [0,1] | \text{there exists } x' \in [0,1] \text{ such that } x - 2x' = V\} \\ &= \{x \in [0,1] | (x - V)/2 \in [0,1]\} \\ &= [0,1] \cap [V, 2 + V],\end{aligned}$$

$$\begin{aligned}\mathcal{X}_k(V) &= \{x \in [0,1] | \text{there exists } x' \in \mathcal{X}_{k-1}(V) \text{ such that } x - 2x' = V\} \\ &= \{x \in [0,1] | (x - V)/2 \in \mathcal{X}_{k-1}(V)\} \\ &= [0,1] \cap \bigcap_{k=1} [(2^k - 1)V, 2^k + (2^k - 1)V],\end{aligned}$$

and

$$\mathcal{X}(V) = \begin{cases} \emptyset & \text{for } V > 0 \text{ or } V < -1, \\ [0,1] & \text{for } -1 \leq V \leq 0. \end{cases}$$

Since for any $V \in [-1, 0]$, for any $x \in X$, $u(x, 1) \leq V$, the value of reconsideration-proof equilibrium is $V = 0$. Any strategies that satisfy for each $t \geq 1$, $\sigma(h^{t-1}) = h_{t-1}^{t-1}/2$ is reconsideration-proof.

Example 4. (The example in Section 5 in Kocherlakota [1996]) Let $u(x, x') = y(x/2 + (1 - x')/(1 + r))$, and $X = [0, 1]$. For a simple exposition, set $y = 1$. Then,

$$\begin{aligned}&\mathcal{X}_1(V) \\ = \;& \{x \in [0,1] | \text{there exists } x' \in [0,1] \text{ such that } x/2 + (1 - x')/(1 + r) = V\} \\ = \;& \{x \in [0,1] | (x/2 - V)(1 + r) + 1 \in [0,1]\} \\ = \;& [0,1] \cap [2(V - 1/(1 + r)), 2V],\end{aligned}$$

$$
\begin{aligned}
& \mathcal{X}_k(V) \\
= \; & \{x \in [0,1] | \text{there exists } x' \in \mathcal{X}_{k-1}(V) \text{ such that } x/2 + (1-x')/(1+r) = V\} \\
= \; & \{x \in [0,1] | (x/2 - V)/(1+r) + 1 \in \mathcal{X}_{k-1}(V)\} \\
= \; & [0,1] \cap \bigcap_{k=1} \left[\left(\left(\frac{2}{1+r}\right)^k - 1 \right) 2\frac{V(1+r)-1}{1-r}, \left(\frac{2}{1+r}\right)^k \left(1 + 2\frac{V(1+r)-1}{1-r}\right) - 2\frac{V(1+r)-1}{1-r} \right],
\end{aligned}
$$

and

$$
\mathcal{X}(V) = \begin{cases} \emptyset & \text{for } V > \frac{1}{1+r} \text{ or } V < \frac{1}{2}, \\ [0,1] & \text{for } \frac{1}{2} \leq V \leq \frac{1}{1+r}. \end{cases}
$$

For any $V \in [\frac{1}{2}, \frac{1}{1+r}]$, for any $x \in X$, $u(x,1) = x/2 \leq V$. Therefore, $V = \frac{1}{1+r}$ is the value of reconsideration-proof equilibrium.

References

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Appendix; Proof of Lemma 1

To prove Lemma 1, the following two lemmas are useful.

Lemma 2. *For any $V \in \mathbb{R}$, for each $k = 0, 1, ...,$ $\mathcal{X}_k(V)$ is closed.*

Proof. The proof is by mathematical induction. Since X is compact and Hausdorff, X is closed. By definition, $\mathcal{X}_0(V)$ is closed. Let k be an arbitrary nonnegative integer. Suppose that $\mathcal{X}_k(V)$ is closed. If $\mathcal{X}_{k+1}(V)$ is empty, the desired conclusion immediately follows. Suppose that $\mathcal{X}_{k+1}(V)$ is nonempty. Take an arbitrary net $\{x_i\}_{i\in I}$ in $\mathcal{X}_{k+1}(V)$ that converges to a point $x \in X$, where I is a directed set. To show that $\mathcal{X}_{k+1}(V)$ is closed, it is sufficient to prove that $x \in \mathcal{X}_{k+1}(V)$. By definition of $\mathcal{X}_{k+1}(V)$, for each $i \in I$, there exists $x'_i \in \mathcal{X}_k(V)$ such that $u(x_i, x'_i) = V$. Then, $\{(x_i, x'_i)\}_{i\in I}$ is a net in $X \times X$. The set $X \times X$ is compact with respect to the product topology by Tychonoff's theorem. Since $X \times X$ is compact, by Theorem 5.2 in Kelley [1975], there exists a subnet $\{(x_{i_j}, x'_{i_j})\}_{j\in J}$ of the net $\{(x_i, x'_i)\}_{i\in I}$ that converges to a point $(\bar{x}, \bar{x}') \in X \times X$. Since the subnet $\{x_{i_j}\}_{j\in J}$ converges to x and $\bar{x}$ and X is Hausdorff, by Theorem 2.3 in Kelley [1975], $x = \bar{x}$. Since the subnet $\{(x_{i_j}, x'_{i_j})\}_{j\in J}$ converges to $(x, \bar{x}') \in X \times X$, for each $j \in J$, $u(x_{i_j}, x'_{i_j}) = V$, and u is continuous, it holds that $u(x, \bar{x}') = V$. We can show that $\bar{x}' \in \mathcal{X}_k(V)$ since $\mathcal{X}_k(V)$ is closed and $\{x'_{i_j}\}_{j\in J}$ converges to $\bar{x}'$. Therefore, $x \in \mathcal{X}_{k+1}(V)$. □

Lemma 3. *For any $V \in \mathbb{R}$, for any $x \in \mathcal{X}(V)$, there exists $x' \in \mathcal{X}(V)$ such that $u(x, x') = V$.*

Proof. Suppose $V \in \mathbb{R}$. If $\mathcal{X}(V)$ is empty, the conclusion trivially follows. Suppose $\mathcal{X}(V)$ is nonempty. Let $x \in \mathcal{X}(V)$. Then, by definition of $\mathcal{X}(V)$, for each $i = 1, 2, ...,$ $x \in \mathcal{X}_i(V)$. By definition of $\mathcal{X}_i(V)$, there exists $x_i \in X$ such that $u(x, x_i) = V$. Since X is compact, by Theorem 5.2 in Kelley [1975], the net $\{x_i\}_{i\geq 1}$ has a convergent subnet $\{x_j\}_{j\in J}$, where J is a directed set. Let $\bar{x} = \lim x_j$. We can show $\bar{x} \in \mathcal{X}(V)$ and $u(x, \bar{x}) = V$ as follows. For each $k = 1, 2, ...,$ a subnet $\{x_j\}_{j\in J, j\geq k}$ is a net in $\mathcal{X}_k(V)$. Since $\mathcal{X}_k(V)$ is a compact set in the Hausdorff space X, by Theorem 5.7 in Kelley [1975], $\mathcal{X}_k(V)$ is closed, and thus $\bar{x} \in \mathcal{X}_k(V)$. Therefore, $\bar{x} \in \mathcal{X}(V) = \bigcap_{k=1}^{\infty} \mathcal{X}_k(V)$. By the continuity of u and Theorem 3.1 in Kelley [1975], $u(x, \bar{x}) = V$ holds. □

Now I can prove Lemma 1.

Proof of Lemma 1. (Sufficiency) Suppose there exists $\{x_t\}_{t\geq 0} \in D(V)$ such that $x_0 = x$. Then, by definition of $D(V)$, for any $t \geq 0$, $u(x_t, x_{t+1}) = V$. This implies that for any $k \geq 1$, $x \in \mathcal{X}_k(V)$. Therefore, $x \in \mathcal{X}(V)$.

(Necessity) Suppose $x \in \mathcal{X}(V)$. By Lemma 3, I can inductively construct a sequence $\{x_t\}_{t\geq 0}$ such that $x_0 = x$ and for each $t \geq 0$, $u(x_t, x_{t+1}) = V$ and $x_{t+1} \in \mathcal{X}(V)$. Clearly, $\{x_t\}_{t\geq 0} \in D(V)$. $\square$