Optimal Intervention in a Random-matching Model of Money*

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Abstract

Wallace [2014] conjectures that there generically exists an inflation-financed transfer scheme that improves welfare over no intervention in pure-currency economies. We investigate this conjecture in the Shi-Trejos-Wright model with different upper bounds on money holdings. The choice of upper bound matters for the optimal policy as some potentially beneficial transfer schemes cannot be studied under small upper bounds. We numerically compute optima for a range of values of the parameters in utility function. Money creation (and accompanying inflation) becomes optimal in more examples when the upper bound on money holdings is larger, and this result is in line with the conjecture.

Keywords: Inflation, Money, Matching Model
JEL classification: E40, E50

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1 Introduction

The Friedman rule has been shown to be optimal in many models of monetary economies. In these models, the rule, typically implemented through lump-sum taxation, improves welfare by reducing the opportunity cost of holding money by raising the return on money . On the other hand, several works have shown that lump-sum transfers financed through inflation can be optimal (Kehoe et al. [1992], Green and Zhou [2005], Molico [2006], and Deviatov [2006] to name a few), even though accompanying inflation lowers the return on money. There, idiosyncratic shocks to endowments or production technology call for risk-sharing, but the frictions inherent in monetary economies inhibit that. As a consequence, lump-sum transfers, which effectively transfer real money balances from the rich to the poor, can improve welfare by providing implicit insurance.

Wallace [2014] argues that such a role of monetary transfer in improving risk-sharing should be a generic feature of models of monetary economies. For money to be a useful social arrangement, models must involve enough imperfect monitoring and discounting so that the folk theorem fails to hold. These frictions give a role to money, but also inhibit risk-sharing at the same time. That observation tells us that policy intervention intrinsically faces the trade-off between raising the return on money as recommended by the Friedman rule and improving risk-sharing.

Based on that, Wallace [2014] conjectures that in a class of economies in which all trade must involve money and there is no explicit taxation due to the frictions, there exist beneficial inflation-financed transfer schemes. The conjecture applies to economies in which trades and policies affect both current-period payoffs and future states of the economy, the typical situation in heterogeneous-agent economies.[1] He discusses two examples, the alternating endowment economy with random "switches" (Levine [1991] and Kehoe et al. [1992]) and a random-matching model due to Shi [1995] and Trejos and Wright [1995], but with a rich set

[1] The conjecture does not apply to the model of Lagos and Wright [2005]. While the degenerated money holding distribution through quasi-linear preferences keeps the model tractable, the degenerated distribution eliminates the role of lump-sum transfer as implicit insurance.

of individual money holdings. However, he presents results only for the former. Here, we present numerical results for a version of the latter model.

In order to do that, we are forced to study a version with a small set of individual money holdings, $\{1, 2, \ldots, B\}$, with two-unit and three-unit of the upper bound on money holdings. Both the discreteness and the bound force us to adapt the policies and the way we model the inflation that results from the transfers. The bound limits transfers to those at the bound. The discreteness forces both transfers and inflation to be probabilistic, where inflation is modeled as a probabilistic version of a proportional tax on money holdings—a tax which is nothing but a normalization when money is divisible. Our main results are for $B = 3$, which, as described below, is mainly dictated by computational feasibility. This magnitude of B is interesting because it is the smallest B that gives potential scope to transfer policies resembling providing interest on money holdings.[2] Deviatov [2006] studies optima under $B = 2$, but the bound of two does not permit the study of regressive transfer schemes. To resemble such transfer schemes, transfers should be given more to those with larger money holdings, but money holdings exceeding one unit are already at the upper bound with $B = 2$. Hence, they cannot receive the transfers. With $B = 3$, we can study transfers that give more transfers to those with larger money holdings.[3]

As in Deviatov [2006], we study alternative steady states in which the planner is choosing the steady-state distribution of money holdings, the trades in meetings subject to those trades being in the pairwise core in each meeting, and the above policies in order to maximize ex ante representative-agent utility. We present results for various combinations of two utility parameters: the discount factor and the finite marginal utility of consumption at zero.

[2]Such policies are called *regressive* schemes in Wallace [2014].

[3]Another approach that can allow a higher upper would be to restrict available policies, as in the model of Jin and Zhu [2017], which is used for studying output response to one-time money injection in a matching environment. There, the planner's transfer scheme is modelled as following. The planner offers a lottery. If an agent pays x, then he will receive $2x$ with probability $\chi > 1/2$, which is a choice of the planner. How many units of lottery (x) an agent buys depends on his money holdings. In the equilibirum, richer agents buy more, so the scheme is said to be endogenously regressive. Thus, the transfer schemes that improves risk-sharing are excluded and some regressive schemes are also excluded. At the cost of the lower upper bounds, we explore optima in a general class of policies.

Consistent with Deviatov [2006], under two-unit upper bound there are few cases in which intervention - inflation-financed government transfer - helps and those interventions are lump-sum transfers. Under three-unit upper bound, more cases have desirable intervention; some are lump-sum and others are transfer that can be interpreted as paying interest on money. We also made attempts to study optima for four-unit upper bound, but we could not get reliable results for all the parameter combinations. Nevertheless, the findings are broadly consistent with the surmise that the set of parameters for which no-intervention is optimal shrinks as the upper bound gets larger.

2 Environment

The environment is borrowed from a random matching model in Shi [1995] and Trejos and Wright [1995]. Time is discrete, and the horizon is infinite. There is a nonatomic measure of infinitely-lived agents. In each period, pairwise meetings for production and consumption occur in the following way. An agent becomes a producer (who meets a random consumer) with probability $\frac{1}{K}$, becomes a consumer (who meets a random producer) with probability $\frac{1}{K}$, or becomes inactive and enters no meeting with probability $1 - \frac{2}{K}$. In a meeting, the producer can produce q units of a consumption good for the consumer in the meeting at the cost of disutility $c(q)$, where c is strictly increasing, convex, and differentiable and $c(0) = 0$. The consumer obtains period utility $u(q)$, where u is strictly increasing, strictly concave, differentiable function on $\mathbb{R}_+$ and satisfies $u(0) = 0$. The consumption good is perishable: it must be consumed in a meeting or discarded. Agents maximize the expected sum of discounted period utilities with discount factor $\beta \in (0, 1)$.

Individual money holdings are restricted to be in $\{0, 1, ..., B\}$. The state of the economy entering a date is a distribution over that set. Then there are pairwise meetings at random at which lottery trades occur: in single-coincidence meetings some amount of output goes from the producer to the consumer and there is a lottery that determines the amount of money that the consumer gives the producer. Next, there are transfers. We let $\tau_i \geq 0$ be the transfer to a person who ends trade with i units and impose only that τ_i is weakly increasing in i for $i \in \{0, 1, ..., B-1\}$ and that $\tau_B = 0$. Finally, inflation occurs via probabilistic disintegration of money. Each unit of money held disappears with probability δ.

We assume that people cannot commit to future actions and that there is no public monitoring in the sense that histories of agents are private. However, we assume that money holdings and consumer-producer status are known within meetings, but that money holdings are private at the transfer stage which is why we assume that τ_i is weakly increasing in i for $i \in \{0, 1, ..., B-1\}$.

All our computations are for $K = 3$, $c(q) = q$, and $u(q) = 1 - e^{-\kappa q}$, which implies that $u'(0) = \kappa$. We study optima for a subset of

$$(\beta, \kappa) \in \{0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.5, 0.6, 0.7, 0.8\} \times \{2, 3, 4, 5, 6, 8, 10, 12, 15, 20\},$$

a subset that satisfies

$$\kappa > 1 + \frac{K(1-\beta)}{\beta}. \tag{1}$$

This condition is necessary and sufficient for the production of constant positive output in a version of the model with perfect monitoring.[4] It is necessary for existence of a monetary equilibrium. The condition leaves fifty-five elements in the subset, and we compute optima for those cases.

[4]The inequality is derived from

$$\frac{d}{dq}_{q=0}\left(-c(q) + \frac{\beta}{K(1-\beta)}\left[u(q) - c(q)\right]\right) > 0$$

Table 1: Variables constituting an allocation

π_k	fraction with k units of money before meetings
$q(k,k')$	production in (k,k') meeting
$\lambda_p^{k,k'}(i)$	probability that producer has i money after (k,k') meeting
$\lambda_c^{k,k'}(i)$	probability that consumer has i money after (k,k') meeting
τ_k	transfer rate for agents with k units of money
δ	probability that money disintegrates after meetings

3 The Planner's Problem

We study allocations that are stationary and symmetric meaning that agents in the same situation (money holdings, producer-consumer status) take the same action. Therefore, productions and monetary payments are constant over all meetings in which a producer has k units of money and a consumer has k' units of money, a (k,k') meeting. A stationary and symmetric allocation consists of the variables listed in Table 1:

The planner chooses production and payment in every meeting, disintegration and transfer rates to maximize ex-ante expected utility before money are assigned according to the stationary distribution. It can be easily shown that ex-ante expected utility is proportional to the expected gains from trade in meetings:

$$\sum_{0\le k\le B}\sum_{0\le k'\le B}\pi_k\pi_{k'}\left[u(q(k,k'))-q(k,k')\right] \tag{2}$$

The planner is subject to the following constraints.

Physical feasibility and stationarity First, money holdings resulting from meetings must be feasible within the pair: in (k,k') meeting, if the consumer has i units, then the producer must have $k+k'-i$ units. Also, money holdings cannot be negative or exceed the total amount brought into the meeting.

$$\lambda_c^{k,k'}(i) = \lambda_p^{k,k'}(k+k'-i) \text{ if } 0 \le i \le k+k' \tag{3}$$

$$\lambda_c^{k,k'}(i) = \lambda_p^{k,k'}(i) = 0 \text{ if } i < 0 \text{ or } k+k' < i \tag{4}$$

Let $\Lambda(k,k')$ denote the set of pairs of probabilities $(\lambda_c^{k,k'}, \lambda_p^{k,k'})$ that satisfy the above constraints.

The money holding distribution is required to be stationary and consistent with transition probability specified by monetary payments in meetings, disintegration and transfer rates. Given a money holding distribution $\{\pi_k\}_{k\in\{0,1,\dots,B\}}$ and a money holding transition probability for each meeting $\{\lambda_p^{k,k'}(i), \lambda_c^{k,k'}(i)\}_{(k,k',i)\in\{0,1,\dots,B\}^3}$, the transition probability that an agent with k units of money before meeting ends up with k' units of money after meeting is

$$t^{(1)}(k,k') = \frac{1}{K}\sum_{i\in\{0,\dots,B\}} \pi_i \left[\lambda_p^{k,i}(k') + \lambda_c^{i,k}(k')\right] + \frac{K-2}{K} 1_{k=k'}.$$

The transition caused by a transfer is expressed by

$$t^{(2)}(k,k') = \begin{cases} 1-\tau_k & \text{if } k<B \text{ and } k'=k, \\ \tau_k & \text{if } k<B \text{ and } k'=k+1, \\ 1 & \text{if } k'=k=B, \\ 0 & \text{otherwise,} \end{cases}$$

and the transition caused by disintegration is expressed by

$$t^{(3)}(k,k') = \begin{cases} \begin{pmatrix} k \\ k' \end{pmatrix} \delta^{k-k'}(1-\delta)^{k'} & \text{if } k \ge k', \\ 0 & \text{otherwise.} \end{cases}$$

Denote $T^{(i)}$ the matrix whose (n, n') elements are $t^{(i)}(n-1, n'-1)$. Specifically,

$$T^{(i)} \equiv \begin{bmatrix} t^{(i)}(0,0) & t^{(i)}(0,1) & \dots \\ t^{(i)}(1,0) & t^{(i)}(1,1) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

The stationarity constraint can be stated as

$$\pi = \pi T^{(1)} T^{(2)} T^{(3)}, \tag{5}$$

where

$$\pi = \begin{bmatrix} \pi_0 & \cdots & \pi_B \end{bmatrix}.$$

Incentive compatibility We assume that agents can deviate individually and also cooperatively from trades, chosen by the planner, in the meeting stage. Individual deviations lead to no trade, and profitable cooperative deviations lead to a Pareto-improving alternative trade. To state incentive compatibility constraints arising from such deviations, it is convenient to use discounted expected utility. Discounted utility for an agent with k units of money before pairwise meeting is denoted by $v(k)$, and that after the meeting stage but before transfer and disintegration is denoted by $w(k)$. They are defined for each symmetric and stationary allocation in a standard way. Specifically, for each $k \in \{0, 1, ..., B\}$,

$$\begin{aligned} v(k) =& \frac{1}{K} \sum_{k' \in \{0,\dots,B\}} \pi_{k'} \left[u(q(k', k)) + \beta \sum_{0 \le i \le k+k'} \lambda_c^{k',k}(i) w(i) \right] \\ &+ \frac{1}{K} \sum_{k' \in \{0,\dots,B\}} \pi_{k'} \left[-q(k, k') + \beta \sum_{0 \le i \le k+k'} \lambda_p^{k,k'}(i) w(i) \right] \\ &+ \frac{K-2}{K} \beta w(k) \end{aligned} \tag{6}$$

$$w(k) = \sum_{k' \in \{0,\dots,B\}} t^{(2)}(k, k') \sum_{i \in \{0,\dots,B\}} t^{(3)}(k', i) v(i) \tag{7}$$

Trades are immune to both individual and cooperative deviations if post-trade allocations are in the pairwise core, and we call this pairwise core constraint. To state the constraint for (k, k') meeting, let $\vartheta(k, k')$ denote a surplus (over no-trade) for a producer in the meeting. The constraint can be stated as follows: $q(k, k')$, $\lambda_p^{k,k'}$, and $\lambda_c^{k,k'}$ solve

$$\begin{aligned}\max_{q\geq 0,\ (\lambda_p,\lambda_c)\in\Lambda(k,k')} & u(q)+\beta\sum_{0\leq i\leq k+k'}\lambda_c(i)w(i)\\ \text{s.t.}\quad & -q+\beta\sum_{0\leq i\leq k+k'}\lambda_p(i)w(i)=\beta w(k)+\vartheta(k,k')\\ & u(q)+\beta\sum_{0\leq i\leq k+k'}\lambda_c(i)w(i)\geq\beta w\left(k'\right)\end{aligned}\tag{8}$$

for some $\vartheta(k, k') \geq 0$.[5] The Karush-Kuhn-Tucker condition is necessary and sufficient for the optimality, and we can derive a set of equations and inequalities from the condition. See Appendix A for the detail.

The planner maximizes the ex-ante expected utility (2), subject to the physical feasibility conditions, the stationarity conditions, and the pairwise core constraints.

4 Results

We compute numerical optima for fifty-five parameter combinations as previously described, for the upper bound of money holdings of two and three (see Appendix B for the computational procedure). Comparing the optima under different upper bound shows how the restriction on money holdings can affect the optimal intervention. Also, we describe some features of optima.

[5]Solving the problem is necessary for trades being in the pairwise core. That is also sufficient if the utility function of the producer and the consumer are strictly monotone in consumption goods and money holdings (see, for example, Mas-Collel et al. [1995]). Here, the utility function may not be strictly increasing in money holdings; Some additional units of money may not be valued in some allocations, and hence the value function w, which specifies the preference for money holdings in trade meetings, may be non-strictly increasing in a part of the domain. In effect, we are solving a relaxed problem using this formulation. For example, if we find that a numerical solution has non-strictly increasing w, it may not be an optimum as solving above problem is not a sufficient condition in that case. It is verified that numerical solutions have strictly increasing w, and thus it is assured that the solutions solve the problem of our interest.

With $B = 2$, we find 2 parameter combinations, $(\kappa, \beta) = (15, 0.7)$ and $(20, 0.6)$, where intervention was optimal. The transfer scheme used in these cases were lump-sum in the sense that they satisfy $\tau_0 = \tau_1$, and the transfer rates were 2.8% and 1.4%. To measure the improvement over no intervention, we compute the amount of consumption that agents are willing to give up to have optimal inflation (and transfer) instead of no-intervention. In particular, we calculate z that satisfies

$$\sum_{k,k'} \pi_k^* \pi_{k'}^* \left[u\left(\frac{100-z}{100} q^*(k,k')\right) - c(q^*(k,k')) \right] = \sum_{k,k'} \pi_k^0 \pi_{k'}^0 \left[u(q^0(k,k')) - c(q^0(k,k')) \right],$$

where q^* is the optimal production with intervention and q^0 is the optimal production with no-intervention. The welfare gain from intervention (z) is 0.08% for $(\kappa, \beta) = (15, 0.7)$ and 0.40% for $(\kappa, \beta) = (20, 0.6)$.

To describe the type of optimal policies, it is helpful to divide them into three groups. With $B = 3$, some intervention was optimal in 21 parameter combinations. Moreover, all interventions were either lump-sum, $\tau_0 = \tau_1 = \tau_2 > 0$, or $\tau_0 = \tau_1 = 0$ and $\tau_2 > 0$. In other words, they turned out to fit the class of transfer scheme discussed in the introduction. Hence, we call the latter type interest-on-money transfer. Because all of the optimum have $\tau_0 = \tau_1$, it is convenient to display them in a table with two numbers, x/y , where x is the common magnitude of τ_0 and τ_1, and y is the magnitude of τ_2. Table 2 reports the optimal transfer rates in such a way, and the measure of welfare gain from intervention (z) is underneath them.

The type of optimal transfer is related to the value of β and κ. Among cases in which some intervention is optimal, the optimal transfer tends to be lump-sum when β and κ are both high, and non-lump-sum when either β or κ is low. It is helpful to spell out the benefits and the costs of the two transfer types to understand this result. The cost for any transfers is the accompanying inflation, as it is lowering the producer's incentive. The benefit of lump-

Table 2: Transfer (%) and welfare gain from intervention (%) , $B = 3$

$\kappa \backslash \beta$	0.15	0.2	0.25	0.3	0.35	0.4	0.5	0.6	0.7	0.8
20	0/0 (0)	0/72.3† (3.52)	0/73.4† (4.92)	0/73.7† (1.87)	0/0 (0)	35.3/35.3* (0.68)	31.2/31.2* (2.04)	13.0/13.0* (6.50)	11.4/11.4* (2.42)	0/0 (0)
15	-	0 /0 (0)	0/65.4† (2.70)	0/66.8† (3.71)	0/67.1† (0.37)	0/0 (0)	0/0 (0)	53.1/53.1* (1.40)	2.8/2.8* (0.48)	0/0 (0)
12	-	-	0/0 (0)	0/59.0† (2.48)	0/61.0† (2.44)	0/0 (0)	0/0 (0)	2.1/2.1* (0.21)	0/0 (0)	0/0 (0)
10	-	-	-	0/0 (0)	0/54.6† (2.52)	0/55.7† (1.05)	0/0 (0)	0/0 (0)	0/0 (0)	0/0 (0)
8	-	-	-	-	0/0 (0)	0/47.2† (1.18)	0/0 (0)	0/0 (0)	0/0 (0)	0/0 (0)
6	-	-	-	-	-	0/0 (0)	0/38.4† (0.58)	0/0 (0)	0/0 (0)	0/0 (0)
5	-	-	-	-	-	-	0/0 (0)	0/0 (0)	0/0 (0)	0/0 (0)
4	-	-	-	-	-	-	-	0/0.6† (0.12)	0/0 (0)	0/0 (0)
3	-	-	-	-	-	-	-	-	0/0.9† (0.18)	0/0 (0)
2	-	-	-	-	-	-	-	-	-	0/0 (0)

Note: $*(\dagger)$ indicates lump-sum (interest-on-money) transfer

sum transfer is the risk-sharing. If an agent without money becomes a consumer, he must forego the opportunity to consume, and this is wasteful from society's point of view. The transfer is helpful in reducing such loss. However, as people get free money from transfer regardless of their money holdings, this type of transfer further lowers producers' incentive to earn money. Hence it tightens producers' participation constraint. In contrast, the benefit of non-lump-sum transfer used in this example, where transfer rate is strictly increasing in an interval, is that it enhances producers' incentive, particularly those who already owns some money. Such transfer can relax the participation constraint of producers who already own some money.

The pattern of optimal transfer fits with the explanation on the benefit and the cost of each transfer type: when people have more incentive to work for future consumption and are more risk averse (that is, when β and κ are both high), the optimal transfer tends to

Table 3: Optimal allocation for $(\kappa, \beta) = (15, 0.7)$

Distribution		Transfer	
π_0	0.346	τ_0	0.028
π_1	0.283	τ_1	0.028
π_2	0.216	τ_2	0.028
π_3	0.155		

production and payment in (k, k') meeting			
$k \backslash k'$	1	2	3
0	0.551/(1)	1.118/(1)	1.396/(1)*
1	0.691/(1)*	0.691/(1)*	0.691/(1)*
2	0.300/(1)*	0.300/(1)*	0.300/(1)*

provide risk-sharing. In contrast, when β and κ are both low, the optimal transfer tends to enhance the incentive of producers. Although there are some examples where no-intervention is optimal with the upper bound of three, we believe that the region where no-intervention is optimal will shrink as the upper bound increases, comparing the optimal transfer rates for $B = 2$ and $B = 3$. The results we attained with the upper bound of four are broadly consistent with this conjecture.

The gain varies from 0.12 percentage point to 6.50 percentage point. The largest gain is attained when the discount factor and the risk aversion are relatively high, and the optimal transfer is lump-sum.

We report details of optima for two examples, $(\kappa, \beta) = (15, 0.7)$ and $(\kappa, \beta) = (15, 0.3)$, in Table 3 and 4. In the former, the optimal transfer is lump-sum, while it goes only to someone with 2 units in the latter. Table 3 shows the money holding distribution π, the transfer τ, and the production and payment in each (k, k') meeting for $(\kappa, \beta) = (15, 0.7)$ and Table 4 shows those for $(\kappa, \beta) = (15, 0.3)$. On the top of each table, the production and payment in meetings are shown in a matrix. Each entry is in the form $q/(\lambda)$, where q is output as a fraction of the first-best level, and λ is the expected monetary payment from the consumer to the producer. A star attached to each entry indicates that the participation constraint for the producer is binding in the meeting.

First, let us compare the distribution. The distribution for $(\kappa, \beta) = (15, 0.7)$ is more

Table 4: Optimal allocation for $(\kappa, \beta) = (15, 0.3)$

Distribution		Transfer	
π_0	0.679	τ_0	0
π_1	0.162	τ_1	0
π_2	0.054	τ_2	0.668
π_3	0.105		

production and payment in (k, k') meeting			
$k \backslash k'$	1	2	3
0	0.207/(1)*	0.305/(2)*	0.305/(2)*
1	0.098/(1)*	0.098/(1)*	0.106/(2)*
2	0/(0)*	0.009/(1)*	0.009/(1)*

concentrated around the center, 1 and 2, achieving more trade meetings than that for $(\kappa, \beta) = (15, 0.3)$. The share of agents with i units of money is decreasing in i in the example for $(\kappa, \beta) = (15, 0.7)$, while $\pi_2 < \pi_3$ holds in the example for $(\kappa, \beta) = (15, 0.3)$ as a result of the non-lump-sum transfer.

The production level is higher in the example for $(\kappa, \beta) = (15, 0.7)$ than in that for $(\kappa, \beta) = (15, 0.3)$ because of more patience. The monetary payment varies from 0 to 2 over meetings in $(\kappa, \beta) = (15, 0.3)$, while it is 1 in all meetings in $(\kappa, \beta) = (15, 0.7)$, achieving the concentrated money holding distribution mentioned above. This result is consistent with an observation in Deviatov [2006].

In the example for $(\kappa, \beta) = (15, 0.7)$, the Individual Rationality constraint for producer is not binding in the $(0, 1)$ and $(0, 2)$ meetings. That means, in these meetings, the terms of trade is not determined by a take-it-or-leave-it offer by the consumer. We find some non-binding IR constraints in all examples in which lum-sum transfer is optimal, while we find all IR constraints are binding in all examples in which non-lump-sum transfer is optimal.

In principle, restricting trading protocol to take-it-or-leave-it offer by the consumer leaves more room for welfare improvements by intervention through transfer. Here, we minimize the necessity of welfare improvements through transfer by allowing any trades in the pairwise core to result in a trade. Our examples show that even in such a setting some intervention can be optimal.

Table 5: Money supply ($\frac{\text{Average money holdings}}{B}$), $B = 2$

$\kappa \backslash \beta$	0.15	0.2	0.25	0.3	0.35	0.4	0.5	0.6	0.7	0.8
20	0.029	0.115	0.181	0.233	0.277	0.319	0.395	0.431	0.451	0.466
15	-	0.039	0.112	0.172	0.223	0.266	0.345	0.413	0.453	0.468
12	-	-	0.051	0.116	0.172	0.220	0.307	0.382	0.449	0.466
10	-	-	-	0.065	0.124	0.177	0.270	0.354	0.429	0.462
8	-	-	-	-	0.060	0.116	0.219	0.313	0.401	0.457
6	-	-	-	-	-	0.028	0.137	0.244	0.352	0.450
5	-	-	-	-	-	-	0.077	0.190	0.307	0.430
4	-	-	-	-	-	-	-	0.113	0.240	0.383
3	-	-	-	-	-	-	-	-	0.127	0.290
2	-	-	-	-	-	-	-	-	-	0.084

Table 6: Money supply ($\frac{\text{Average money holdings}}{B}$), $B = 3$

$\kappa \backslash \beta$	0.15	0.2	0.25	0.3	0.35	0.4	0.5	0.6	0.7	0.8
20	0.029	0.138	0.206	0.255	0.258	0.281	0.319	0.341	0.384	0.444
15	-	0.041	0.133	0.195	0.243	0.241	0.292	0.335	0.393	0.436
12	-	-	0.054	0.136	0.194	0.194	0.270	0.329	0.384	0.429
10	-	-	-	0.070	0.143	0.197	0.242	0.324	0.361	0.423
8	-	-	-	-	0.064	0.133	0.194	0.282	0.336	0.412
6	-	-	-	-	-	0.031	0.154	0.215	0.302	0.392
5	-	-	-	-	-	-	0.087	0.165	0.275	0.362
4	-	-	-	-	-	-	-	0.128	0.208	0.317
3	-	-	-	-	-	-	-	-	0.147	0.256
2	-	-	-	-	-	-	-	-	-	0.101

We report other features of optima in the following. As expected, welfare at the optimum is increasing with the discount factor. The welfare at optimum appears to increase with B as well. Related to this result, Zhu [2003] shows that the set of implementable allocations for lower n is a subset of the set for larger n when B increases as in $B_n = m^n$ for any integer $m > 1$. Hence, welfare is a weakly increasing function for n if B_n increases in that way. However, it is not straightforward whether the result will extend to $B_n = n$ for $n = 1, 2, \ldots$. At least to our knowledge, numerical results that compared welfare at the optimum for different B (greater than 1) do not exist. Our result is in line with the hypothesis of increasing welfare at optimum in B.

We define money supply as the average money holdings of people relative to the upper bound. Table 5 and 5 report the money supply for $B = 2$ and $B = 3$ respectively. Optimal money supply monotonically increases with discount factor and risk aversion. Comparing two upper bounds, optimal money supply is less volatile with the higher upper bound. This is natural as the wealth level can be more finely recorded with the richer money holdings. It does not exceed a half of B in any cases.

The lottery is potentially useful as money is indivisible and money holding is limited (due to the concavity of utility function, it is not helpful to use a lottery for production level). We computed the percentage of trade meetings where the lottery is used to any degree to find how frequently the lottery is used. It tends to be used more with high discount rate and risk aversion but is not monotonically increasing. As the value of money at optimum increases with the discount rate, the lottery gets used more often to overcome the indivisibility.

We construct a similar measure for the trade meetings where consumers are making a take-it-or-leave-it offer. As this trading mechanism is popular for its simplicity, it is interesting to see whether such trading mechanism is close to the optimal one. Although our measure does not capture the welfare loss from imposing a take-it-or-leave-it trading mechanism, it is indicative. In one example, almost 50% of meetings are departing from a take-it-or-leave-it trading mechanism. Again, this is observed when both parameters are high. Considering the optimum with $B = 1$ sheds some light on the reason why the departing if salient with a high discount rate. When the upper bound is one and discount rate is sufficiently high, the optimum is achievable by making the half of population holding money and producers producing the first-best level of production. Consumers don't get all the surplus in that case, as it will make producers to produce too much. The result here shows the similar pattern, that is, consumers don't get all the surplus when the discount rate is high.

5 Concluding Remarks

We computed optima for a range of utility parameters under two-unit and three-unit upper bound on money holdings. The scope of transfer meaningfully expands as the upper bound changes from two-unit to three-unit, as three-unit is the lowest one that makes interventions paying interests on money feasible. We found that such interventions are optimal under some utility parameter specifications, while lump-sum transfers or no intervention were optimal in other specifications. Hence, both types of interventions can be optimal depending on the model parameters, and it may not be innocuous to restrict focus to only one type. Our result loosely confirms a version of the conjecture of Wallace [2014]: the number of parameter sets under which no intervention is optimal decreases from the two-unit bound case to the three-unit case, and the optimal intervention depends on the model details. We say loose because we studied limited numbers of utility parameter and upper bound.

Lastly, we would like to comment on the examples where no-intervention is optimal. While our results may appear to be inconsistent with the conjecture in Wallace [2014], we believe that it is due to the small bounds on the money holdings. The transfer policies and the accompanying inflation are modelled as probabilistic changes on one's money holdings, and that give rise to the additional uncertainty. Such uncertainty would disappear with a larger upper bound. In that sense, the results are consistent with the conjecture. The results are also consistent with other aspects of the conjecture in Wallace [2014]. Although we allow the class of policies that are richer than Wallace [2014], the optimal policies are in the class of the conjecture in Wallace [2014].

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A Pairwise Core Constraint

In this appendix, we characterize the solution for 8. The problem (8) can be rewritten as

$$
\begin{aligned}
&\max_{\lambda_c} u\left(\beta \sum_{0\le i\le k+k'} \lambda_c(i) w\left(k+k'-i\right) - \beta w(k) - \vartheta(k,k')\right) + \beta \sum_{0\le i\le k+k'} \lambda_c(i) w(i) \\
&\text{s.t. } -\left(\beta \sum_{0\le i\le k+k'} \lambda_c(i) w\left(k+k'-i\right) - \beta w(k) - \vartheta(k,k')\right) \le 0 \\
&\qquad -\lambda_c(i) \le 0, \text{ for all } i \\
&\qquad 1 - \sum_{0\le i\le k+k'} \lambda_c(i) = 0
\end{aligned}
$$

Let $\mu_q^{k,k'}, \mu_0^{k,k'}(i), \mu_{\text{sum}}^{k,k'}$ denote the multipliers for the constraints. Since all constraints are linear, the constraint qualification is satisfied. As the objective function is concave, the Karush-Kuhn-Tucker (KKT) condition is necessary and sufficient for the optimality. The KKT condition can be stated as follows: for some $\vartheta(k,k') \ge 0$,

$$0 = \left[u'\left(q(k,k')\right) + \mu_q^{k,k'}\right] \beta w\left(k+k'-i\right) + \beta w\left(i\right) + \mu_0^{k,k'}(i) + \mu_{\text{sum}}^{k,k'} \text{ for all } i, \tag{9}$$

$$0 \ge -q(k,k'), \tag{10}$$

$$0 \ge -\lambda_c^{k,k'}(i) \text{ for all } i, \tag{11}$$

$$0 = 1 - \sum_{0\le i\le k+k'} \lambda_c^{k,k'}(i), \tag{12}$$

$$0 = \mu_q^{k,k'} q(k,k'), \tag{13}$$

$$0 = \mu_0^{k,k'}(i) \lambda_c^{k,k'}(i) \text{ for all } i, \tag{14}$$

and

$$q(k,k') \quad = \quad \beta \sum_{0\le i\le k+k'} \lambda_c^{k,k'}(i) w(k+k'-i) - \beta w(k) - \vartheta(k,k').$$

Note that the surplus variable $\vartheta(k,k')$ appears only in the last condition. The existence of $\vartheta(k,k') \geq 0$ which satisfies the last condition is equivalent to the individual rationality constraint

$$-q(k,k') + \beta \sum_{0 \leq i \leq k+k'} \lambda_c^{k,k'}(i) w(k+k'-i) \geq \beta w(k). \tag{15}$$

Therefore, the KKT condition can be expressed by (9-14) and (15), without $\vartheta(k,k')$.

B Computational Procedure

We compute solutions for the planner's problem using two solvers that are compatible with the GAMS interface, KNITRO and BARON. KNITRO is a local solver for large-scale optimization problems. For a given initial point, it quickly converges to a local solution (or shows that it cannot reach one), but it does not guarantee global optimality. This issue is usually dealt with by using a large number of initial values. The solver automatically feeds in different initial values as we change an option that controls the number of initial values. In contrast, BARON (Branch-And-Reduce Optimization Navigator) is a global solver for nonconvex optimization problems. It continues to update an upper bound and a lower bound on the objective by evaluating the values of variables satisfying the constraints and stops when the difference between the two bounds becomes smaller than a threshold. It guarantees global optimality under mild conditions, but it tends to take much longer time to converge than local solvers. Even before it converges, we can terminate it and see its candidate solution. When Baron did not finish in a reasonable time span, we stopped it and checked the candidate solution with the solution from KNITRO.

For $B = 2$, BARON under its default criterion usually finished in about an hour, and it reproduces the solution that KNITRO finds using 250 different initial points. For $B = 3$, BARON did not finish in 200 hours. In all cases that we computed, the candidate solution was not updated after roughly 20 hours. (The remaining time was being used to verify that other feasible allocations are not better than the candidate solution.) We ran KNITRO with

1000 initial points and found that its solution coincides with the intermediate output from BARON, which is the best lower bound. Also, to check whether the computation is sensitive to the number of initial points we use, we ran KNITRO with 8000 initial points and made sure that the results are the same.

We also tried the same approach with $B = 4$, but we could not find robust results for some examples. For $B = 4$, the risk aversion parameter κ is varied over $\{2, 3, 4, 5, 6, 8, 10, 12, 15, 20\}$ and the discount factor parameter β is varied over $\{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$. Optimum is computed for each of 46 pairs of κ and β, under which the condition (1) is satisfied, out of 70 possible pairs, using (i) KNITRO with 8000 initial points, (ii) KNITRO with 16000 initial points, or (iii) BARON.

We found the numerical results for $B = 4$ are less reliable than those for $B = 2$ or 3: In roughly a third of the cases, either KNITRO gives different answers depending on the number of initial conditions or KNITRO and BARON give different answers.